THE UNRELATED QUESTION RANDOMIZED RESPONSE MODEL[1/]

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I. Introduction

Warner[2/] has developed a randomized response technique which allows the respondents in personal interview surveys to provide information on sensitive or highly personal questions and yet retain their privacy. The technique requires the respondent to select one of two related questions using a random device. The respondent answers only "yes" or "no" to the chosen question without revealing which question was actually selected. The responses to either question divide the population into the same two mutually exclusive and complementary classes. The proportion of "yes" answers in the sample and the known chance of selecting either question is sufficient to provide an unbiased estimate of the proportion of the population in each of the mutually exclusive classes, provided the respondents answer truthfully.

A variation of the Warner technique, suggested by Walt R. Simmons and designed to increase further the cooperation of the respondents and the veracity of their responses, is reported in this paper. It requires the respondents to randomly select one of two unrelated questions, so that the mutually exclusive and complementary properties of the Warner technique no longer apply. Two samples are required with a different set of selection probabilities for the two questions for each sample. The method for estimating the parameters and variances for this alternative randomized response model are developed also for two independent trials per respondent. Results from two empirical studies concerned with estimating the proportion of illegitimate births from household interviews are reported.

II. The Warner Randomized Response Model

The purpose of the Warner Model is to provide a method for estimating the proportion of persons with a sensitive attribute, say A, without requiring the individual respondent to report his classification (whether it be A or $\bar{A}$) to the interviewer. The respondent is provided with a random device to choose one of two statements of the form:

1. "You have the attribute A"
2. "You do not have the attribute A"

Without revealing to the interviewer which statement has been chosen, the respondent then answers "yes" or "no" according to the statement he has selected and whether he does or does not have the attribute A.

Let

Π = true proportion with attribute A

p = probability that the first statement is selected (the second statement is selected with probability $1-p$)

x_i = 1 if the i-th respondent says "yes" to the selected statement

x_i = 0, otherwise

n = sample size

Then, with a single sample and a single trial with respondents who always tell the truth,

$$\Pr(x_i=1) = \Pi p + (1-\Pi)(1-p)$$

$$\Pr(x_i=0) = (1-\Pi)p + (1-p)\Pi$$

It follows that the maximum likelihood estimate of Π is

$$\hat{\Pi} = \frac{p-1}{2p-1} + \frac{n_1}{n(2p-1)}, \quad p \neq \tfrac{1}{2}$$

where $n_1 = \sum_{i=1}^{n} x_i$. This is an unbiased estimate, if all respondents answer truthfully, with variance given by

$$\mathrm{Var}(\hat{\Pi}) = \frac{\Pi(1-\Pi)}{n} + \frac{p(1-p)}{n(2p-1)^2}$$

III. The Simmons Unrelated Question Randomized Response Model[3/]

The Warner technique is designed to elicit truthful answers to questions many respondents would refuse to answer at all, if asked directly. Walt R. Simmons has suggested that the confidence of the respondents might be further increased and hence the likelihood of truthful answers, if two unrelated questions (or statements) are used, one pertaining to the sensitive attribute, say A, and the other to a non-sensitive characteristic, say B.

1/ Research supported by the National Center for Health Statistics, Contract No. PH 86-65-68.

2/ S. L. Warner, "Randomized Response: A Survey Technique for Eliminating Evasive Answer Bias," Journal of the American Statistical Association, 60, (1965), pp. 53-69.

3/ The Simmons single trial unrelated question model has been discussed in some detail by Abdel-Latif A. Abul-Ela, "Randomized Response Models for Sample Surveys on Human Populations," unpublished Ph.D. thesis, University of North Carolina, Chapel Hill, 1966.

It is noted that both of the Warner statements divide the population into the same two mutually exclusive and complementary classes. The unrelated question model uses two statements of the form:

1. "You have the attribute A "
2. "You have the attribute B "

so that some respondents might belong to both groups and some might not belong to either group.

Two independent samples are required with this model. Let

Π_1 = true proportion with sensitive attribute A

Π_2 = true proportion with non-sensitive attribute B (not related to A)

p_1 = probability that the first statement is selected by each respondent in the first sample (the second statement is selected with probability $1-p_1$ by respondents in this sample)

p_2 = probability that the first statement is selected by each respondent in the second sample, $p_2 \neq p_1$

x_i = 1 if the i-th respondent in the first sample says "yes" to the selected statement

= 0, otherwise

y_i = 1 if the i-th respondent in the second sample says "yes" to the selected statement

= 0, otherwise

n_1 = size of the first sample

n_2 = size of the second sample

With a single trial per respondent,

$$\Pr(x_i=1) = p_1\Pi_1 + (1-p_1)\Pi_2$$

$$\Pr(y_i=1) = p_2\Pi_1 + (1-p_2)\Pi_2 \ ,$$

provided all respondents answer truthfully. If $n_{11} = \sum_{i=1}^{n_1} x_i$ and $n_{12} = \sum_{i=1}^{n_2} y_i$ are the respective number of "yes" answers in the two samples, then unbiased estimates of Π_1 and Π_2 may be obtained by solving the pair of equations (i.e. by equating observed proportions of "yes" answers to expected proportions)

$$\frac{n_{11}}{n_1} = p_1\Pi_1 + (1-p_1)\Pi_2$$

$$\frac{n_{12}}{n_2} = p_2\Pi_1 + (1-p_2)\Pi_2 \quad .$$

The estimates are:

$$\hat{\Pi}_1 = \frac{1}{p_1-p_2}\{(1-p_2)(n_{11}/n_1)-(1-p_1)(n_{12}/n_2)\}$$

$$\hat{\Pi}_2 = \frac{1}{p_2-p_1}\{p_2(n_{11}/n_1) - p_1(n_{12}/n_2)\}$$

It is noted that if Π_2 is known, then a single sample is sufficient to estimate Π_1 . The estimator is

$$\hat{\Pi}_1 = \frac{(n_{11}/n_1) - (1-p_1)\Pi_2}{p_1}$$

IV. Test of the Simmons Unrelated Question Model (Single Trial Per Respondent)

The Simmons Model was tested in late October 1965 by personal interviews in a total sample of 148 households in which it was known that a birth had occurred during August and September 1965. The sample was selected from birth certificates upon which the marital status of the mother on the date of the birth was recorded. Twenty-eight (28) or 18.9 percent of the 148 mothers were not married. The respondents were asked to select a card from a shuffled deck of 50 cards and to answer "yes" or "no" to the truth of the statement printed on the card. The two statements used in the deck were:

1. "There was a baby born in this household after January 1, 1965, to an unmarried woman who was living here." (Attribute A)
2. "I was born in North Carolina." (Attribute B)

The results obtained in this test of the technique and model are shown in Table 1. The estimated proportion of all households with a birth to an unmarried woman, that is $\hat{\Pi}_1$, is in reasonable agreement with the true proportion. The results are even closer when computed separately for white and non-white households.

It is reasonable to ask whether results as good or better could have been obtained by direct questioning of the respondents. Although this has not been tested, the completeness with which births known to have occurred out of wedlock are reported in household interviews has been found to be somewhat less than for births classified as legitimate on the birth certificate.[4] The latter results indicate that the legitimacy status of births is sufficiently sensitive to warrant use of a technique which respects privacy of the respondent.

Table 1

Parameters and Estimates in Test of Simmons Model (Single Trial per Respondent)

Item	All Households	White Households	Nonwhite Households
p_1	.7	.7	.7
p_2	.3	.3	.3
n_1	63	40	23
n_2	85	64	21
n_{11}	24	10	14
n_{12}	49	31	18
$\hat{\Pi}_1$	.235	.074	.423
Π_1	.189	.077	.454
$\hat{\Pi}_2$	.722	.660	1.043

V. Extension of Simmons Model to Two Trials per Respondent

A simple extension of the Simmons unrelated question model requires each respondent to make two independent selections of the two questions (or statements) using the randomizing device. Let n_{11}, n_{10}, n_{01}, n_{00} be the numbers of individuals answering (Yes, Yes), (Yes, No), (No, Yes) and (No, No) respectively in the first sample and m_{11}, m_{10}, m_{01} and m_{00} be the corresponding numbers for the second sample. The sample sizes are n_1 and n_2. As before the observed proportions of "yes" answers for each sample are equated to the expected values of these proportions and the unknown parameters Π_1 and Π_2 are estimated by solving the resulting pair of equations,

$$\gamma_1 = \frac{2n_{11} + n_{10} + n_{01}}{2n_1} = p_1\Pi_1 + (1-p_1)\Pi_2$$

$$\gamma_2 = \frac{2m_{11} + m_{10} + m_{01}}{2n_2} = p_2\Pi_1 + (1-p_2)\Pi_2$$

Thus,

$$\hat{\Pi}_1 = \frac{1}{p_1-p_2} \{(1-p_2)\,\gamma_1 - (1-p_1)\,\gamma_2\}$$

$$\hat{\Pi}_2 = \frac{1}{p_2-p_1} \{p_2\gamma_1 - p_1\gamma_2\}$$

These are unbiased estimates if all respondents answer truthfully.

VI. Test of the Simmons Model With Two Trials Per Respondent

Two different randomizing devices were used with stratified cluster samples of North Carolina households in the summer of 1967 to test the two trial model. The first device was the same as used previously, a deck of 50 cards. The second was a sealed plastic box, designed by B. G. Greenberg, containing red and blue beads.[5] When the box is shaken by the respondent, a bead appears in a small window and the respondent answers "yes" or "no" to the statement corresponding to the color of the bead. Both statements are attached to the box. The two statements used with both randomizing devices were:

1. "In the past 12 months there was a baby born in this household to an unmarried woman who was living here at the time."
2. "I was born in North Carolina."

The interview procedure for the deck of 50 cards randomizing device is given in Appendix A. The parameters, distributions of "yes" and "no" answers, and estimates for the respective randomizing devices are given in Tables 2 and 3. In contrast to the first field trial, the estimated Π_1's for both devices bear little relationship to the values expected for this parameter.

VII. Some Speculations

Under the basic unrelated question model, the major field trials for North Carolina yielded as the estimate of the proportion of households with an illegitimate birth, the value $\hat{\Pi}_1 = 0.14$. Vital records for the State indicate a figure $\Pi_1 = 0.008$. Thus the estimate is almost 20 times the criterion value. Why did the basic model fail so badly? This question has even more force when it is recalled that the same model was highly successful in the first (pilot) test, and further

4/ Horvitz, D. G. "Problems in Designing Interview Surveys to Measure Population Growth." Proceedings, Social Statistics Section, American Statistical Association, (1966), 245-249.

5/ The Department of Biostatistics, University of North Carolina School of Public Health, supplied the plastic boxes and beads for this trial.

Table 2

Parameters and Estimates in Test of Unrelated Question Model with Two Trials per Respondent Randomizing Device: Deck of 50 Cards

Item	All Households	White Households	Nonwhite Households	All Households Reporting Births
p_1	.7	.7	.7	.7
p_2	.3	.3	.3	.3
n_1	1450	1227	223	88
n_2	1638	1340	298	103
n_{11}	166	137	29	17
n_{10}	323	271	52	16
n_{01}	298	253	45	18
n_{00}	663	566	97	37
m_{11}	636	512	124	47
m_{10}	345	291	54	25
m_{01}	276	215	61	13
m_{00}	381	322	59	18
γ_1	.328	.325	.348	.379
γ_2	.578	.571	.609	.641
$\hat{\Pi}_1$	.142	.141	.151	.183
Expected Π_1	.008	.002	.034	.042*
$\hat{\Pi}_2$	.765	.755	.805	.837

*Based on data from matched birth certificates.

Table 3

Parameters and Estimates in Test of Unrelated Question Model with Two Trials per Respondent Randomizing Device: Plastic Box of 50 Beads

Item	All Households	White Households	Nonwhite Households	All Households Reporting Births
p_1	.7	.7	.7	.7
p_2	.3	.3	.3	.3
n_1	437	320	117	29
n_2	442	375	67	28
n_{11}	53	37	16	4
n_{10}	76	55	21	6
n_{01}	83	61	22	2
n_{00}	225	167	58	17
m_{11}	166	141	25	6
m_{10}	80	67	13	8
m_{01}	53	48	5	3
m_{00}	143	119	24	11
γ_1	.303	.297	.321	.276
γ_2	.526	.529	.507	.411
$\hat{\Pi}_1$	.136	.122	.180	.174
Expected Π_1	.008	.002	.034	.080*
$\hat{\Pi}_2$	.693	.704	.648	.512

*Based on data from matched birth certificates.

that the direction of the failure might seem on first inspection to imply that far too many--rather than possibly too few--persons replied "yes" to the question of whether there had been an illegitimate birth in their household.

Let it be noted immediately that we do not claim to know the answer to this question. But study of the model and the experiment suggest a number of possibilities, and several significant relevant facts. A review of some of these matters makes it clear that the range and variety of possible hazards is great. It also suggests several types of modification in the basic model or its application to protect against selected hazards. First, one must consider the possibility of a boner in the coding or processing of data. That source of failure is consistent--as are several other hypotheses--with the observation that the scale of error is so very large, and is fairly constant over nearly all of 8 different subsamples, and six different domains of study. The error in result is persistently constant. One can never be absolutely certain that all boners have been removed from data reduction, but diligent audit has failed to uncover residual flaws of that type.

One very important class of causes of model failure is that characterized by the realized p-values being different from the intended probabilities. That is, the actual proportion of respondents being called upon to answer Statement 1 may be different from the *a priori* probability that Statement 1 will be drawn. Within this class are found several quite separate kinds of possible circumstances. (We'll speak in terms of the decks of cards, although similar remarks might be made for the plastic box randomizing device.) It could be that the deck as used contained, say, 60 percent of the cards with Statement 1 rather than the intended 70 percent--and this could have occurred either because the deck was initially made up incorrectly, or because it changed constitution during the trials, having been dropped or otherwise acquiring an imperfectly operating state or condition. If, for example, Statement 1 had been answered 61% of the time rather than the expected 70% in the first sample (green deck), or 32% rather than 30% in the second sample (yellow deck), the basic model would yield almost perfect results. (The sampling standard error for p is approximately 0.012.)

The deck may have had correct proportions, and the drawing have been truly random, but the effective p-values still incorrect because of respondent reaction. One hypothesis might be that some proportion of respondents was sufficiently confused by the game--or not in sympathy with it--that they made up their mind that they would answer "yes" in any event, and so in fact, among those who drew Statement 1, the effective proportion answering Statement 1 was less than the proportion drawing the statement, the remainder answering no particular question, but just saying "yes." A number of plausible variations of this hypothesis are easily formed.

Another class of possible causes of failure might be termed response errors. This class, too, is broad and heterogeneous. One member of the class is misreading of Statement 1 by the respondent. The statement is

> "In the past 12 months there was a baby born in this household to an unmarried woman who was living here at the time."

It's not hard to imagine that in the mind of some readers the unmarried gets translated married, leading to a corresponding gross overstatement of "yes." It is possible, too, that the manner of the interviewer in presenting or clarifying the game to the respondent, or especially to those respondents who had concern or difficulty in replying, contained a bias that encouraged a "yes" answer. Another type of response error is conscious non-truthful reply by respondents. If the end result had been too few "yeses," this would have been a prime contender for the villain's role. But with the opposite factual result, deliberate untruthfulness appears less likely.

The problem was more difficult in the Statewide study than in the pilot study, because in the latter the true Π_1 was about 0.2, while in the former the true Π_1 was less than one percent--nearly zero. Contrastingly, and for a good reason, Π_2 was very much larger, and thus any operating deviation from expectation, even though slight, had a most substantial relative impact on estimating the near-zero Π_1. This is simply a case of the well-known fact that it is difficult to estimate near-zero parameters by sampling procedures. (We admit with embarrassment that we mistakenly thought in planning the study that the true Π_1 was of the order of 0.08 rather than 0.008. This "boner" arose because we were thinking of the proportion of births which were illegitimate rather than the proportion of households with an illegitimate birth in one year.)

When the basic model was formulated, an unrelated question was chosen for which it was possible to secure a criterion measure from an outside source. The reason for the choice was that use of the outside criterion would permit a modification in the model which might make it more efficient, or alternatively provide a means for validating results. At this point the outside source is mentioned only because we wish to note that some of the alternate models considered are quite sensitive to any error in the outside source and accordingly, it is necessary that the external data be true to within close tolerances if they are to be used.

We turn now to the observation that there are many possible alternate models which are more or less closely allied to the basic 2-deck 2-trial model discussed in Section V above. Some of them are consistent with one or the other of the speculations just offered. Indeed there are a number of these alternative models which are plausible. One example is presented in Section VIII below. Unfortunately we have as yet been unable to explore quantitatively other models. But in Section IX we call attention to several promising routes which illustrate types of models that deserve investigation.

VIII. Alternate Model I

This model assumes that the realized values of p_1 and p_2 are not as expected but are modified by a factor λ. This could occur for any number of reasons; e.g. a certain proportion of the respondents who actually have attribute A and select the statement referring to attribute A respond instead to their status concerning attribute B. The equations become

$$\gamma_1 = \lambda p_1 \Pi_1 + (1-\lambda p_1)\Pi_2$$

$$\gamma_2 = \lambda p_2 \Pi_1 + (1-\lambda p_2)\Pi_2 ,$$

and can be rearranged to show that

$$\lambda(\Pi_1 - \Pi_2) = \frac{\gamma_1 - \Pi_2}{p_1} = \frac{\gamma_2 - \Pi_2}{p_2} ,$$

yielding a solution for Π_1 in terms of λ and Π_2 :

$$\Pi_1 = \Pi_2 - \frac{\Pi_2 - \gamma_1}{\lambda p_1} ,$$

or

$$\Pi_1 = \Pi_2 - \frac{\Pi_2 - \gamma_2}{\lambda p_2} .$$

The estimator for Π_2 is the same as before, namely:

$$\hat{\Pi}_2 = \frac{p_2\gamma_1 - p_1\gamma_2}{p_2 - p_1} .$$

It is assumed next that $\Pi_1 = 0$ in households not reporting any births. When Π_1 does equal zero, and data for these households are used, an estimate for λ is obtained:

$$\hat{\lambda} = \frac{\hat{\Pi}_2 - \gamma_1}{\hat{\Pi}_2 p_1} ,$$

or

$$\hat{\lambda} = \frac{\hat{\Pi}_2 - \gamma_2}{\hat{\Pi}_2 p_2} .$$

Finally, using these values, Π_1 is estimated with

$$\hat{\Pi}_1 = \hat{\Pi}_2 - \frac{\hat{\Pi}_2 - \gamma_1}{\hat{\lambda} p_1} .$$

Results from this model are reported in Tables 4 and 5. The estimates of Π_1 in these tables are in fairly close agreement with the values expected. The estimates of the adjustment factor λ are all in the neighborhood of .82 for the various household groups. Negative estimates occurred in several instances for which the expected Π_1 is close to zero.

Since this model sets $\Pi_1 = 0$ for households not reporting births, and these are the vast majority of the households, it can be argued that the estimated Π_1 for <u>all</u> households will be forced to be close to zero. Despite this, the model behaved rather well yielding sensible estimates of Π_1 for white versus nonwhite households and for households reporting births.

IX. Outline of Other Alternate Models

We note here in outline only several types of alternative models which deserve further investigation. They may suggest still additional ways of utilizing the kind of information obtained in the North Carolina experiment.

<u>Alternate II</u>. Same as Alternate I except that adjustment to the p values is made in an additive rather than multiplicative form.

<u>Alternate III</u>. Utilizing the data in the yes-yes, yes-no, no-yes, and no-no cells, it is possible to set up equations with λ , Π_1 and Π_2 as the unknowns, and to solve simultaneously for all three parameters, using <u>all</u> the data (rather than just the no-birth households for estimating λ).

<u>Alternate IV</u>. Assuming the realized values of p_1 and p_2 are unknown and again using the data in the yes-yes, yes-no, no-yes and no-no cells, it is possible to solve for the four parameters p_1 , p_2 , Π_1 and Π_2 using an iterative procedure.

<u>Alternate V</u>. An estimate of Π_2 (proportion of respondents born in North Carolina) can be secured from an external source. Using that value, and the domain of households with no

Table 4

Parameters and Estimates Obtained with Alternate Model I

Item	All Households	White Households	Nonwhite Households	All Households Reporting Births
	Randomizing Device: Deck of 50 Cards			
$\hat{\lambda}$	.817	.813	.840	.816
$\hat{\Pi}_1$	.002	-.0003	.027	.037
Expected Π_1	.008	.002	.034	.042*
	Randomizing Device: Plastic Box of 50 Beads			
$\hat{\lambda}$	.811	.825	.745	.811
$\hat{\Pi}_1$	.006	-.0007	.020	.096
Expected Π_1	.008	.002	.034	.080*

* Based on data from matched birth certificates.

Table 5

Parameters and Estimates Obtained with Alternate Model I

Combined Data*

Item	All Households Reporting Births	All White Households Reporting Births	All Nonwhite Households Reporting Births
$\hat{\lambda}$	.816	.815	.829
$\hat{\Pi}_1$	.041	-.010	.123
Expected Π_1	.051**	.019**	.140**
$\hat{\Pi}_2$	.770	.711	.990
n_1	117	80	37
n_2	131	100	31

* Data for both randomizing devices were pooled for these estimates.

** Based on data from matched birth certificates.

births, derived values of realized p's can be obtained and then a solution for Π_1 for each of the two samples, the latter being averaged to obtain a final estimate for Π_1.

Alternate VI. Using a known value of Π_2 from the external source, it is possible to solve the simultaneous system for unknown values of p_1 and p_2 in terms of Π_1, and then a resulting quadratic equation for Π_1.

In closing we should like to emphasize our belief that the randomized response method includes a very large family of techniques, which are just beginning to be recognized and explored.

APPENDIX A: INTERVIEW PROCEDURE FOR TWO TRIALS PER RESPONDENT

G. Randomized Card Question

(READ THE FOLLOWING STATEMENT TO THE RESPONDENT, BUT DO NOT HAND CARDS TO RESPONDENT UNTIL YOU HAVE FINISHED)

"Here are some cards, each of which has a statement on it. There are just two different statements. Each statement can be answered simply as 'true' or 'false.' You are to answer only one of the statements by picking a card from the deck at random. We are using this procedure so that I will <u>not</u> know which of the two statements you happen to select. Since both of the statements would be true for some people and false for others, I will not be able to tell from your answer which one <u>you</u> have chosen.

The cards with the first statement are marked with a circle and the cards with the second statement are marked with a square. When I hand you the cards you may look through them and read the statements if you wish. Then, shuffle the cards thoroughly and, <u>without looking</u>, select one of them, being careful not to show it to me. Simply answer 'true' or 'false' to the statement on the card you happen to select. Replace the card in the deck, shuffle the cards, and hand them back to me. (HAND CARDS TO RESPONDENT) (RECORD RESPONSE TO FIRST TRIAL)

Now that you fully understand how the game works, let's play it just one more time. Forget about the question you have just answered. Please shuffle the cards another time and select one of them, without looking or showing the card to me. Again, simply answer 'true' or 'false' to the statement you select, shuffle the cards and hand them back to me." (HAND CARDS BACK TO RESPONDENT. RECORD ANSWER TO SECOND TRIAL.)

G1. First Reply: True ☐ False ☐ Refused ☐

G2. Second Reply: True ☐ False ☐ Refused ☐

G3. Color of random card set used: Green ☐ Yellow ☐

G4. Respondent's name: ______________________ Time interview ended: __________